Model Answer

AS-2115

M.Sc. (First semester) Examination, 2013

PHYSICS

CLASSICAL MECHANICS

Maximum Marks : 60

Section-A $10 \times 2 = 20$

1. Choose the correct answer!

(i) (b) (ii) b (iii) b (iv) (a) (v) (b) (vi) (b) (vii) (c) (viii) (b) (ix) (c) (x) (b)

Section-B

2. What are constraints? Explain their different type and classes by giving examples.

Ans:- Constraints - Any limitations or geometrical restrictions on the motion of the system of particles are known as constraints.

Constraints are classified into different types and classes based on four criteria namely, (i) Whether they time dependent or time independent (ii) Whether they are integrable algebric relation among the co-ordinates or non-integrable ones, (iii) Whether they are conservative or dissipative, and (iv) whether they are algebric equation or algebric inequalities.

(1.) Holonomic and non-holonomic Constraints:-

A holonomic constraint is one that may be expressed in the form of an equation relating the co-ordinates of the system and time. The general form of such equation for a system of N-particles is

$$f_i(x_1, y_1, z_1, x_2, y_2, z_2, \ldots \ldots x_N, y_N, z_N, t) = 0$$

where $i = 1, 2, 3, \ldots k$ (denotes the constraints) and f_i is some function of the co-ordinates and independent of velocities. Example of holonomic constraint is the motion of a rigid body; which is require here to explain in detail.

A non-holonomic constraint cannot be expressed in the form of above equation. It may be in the form of inequality. Moreover, it may also depend on the velocity. The motion of the gas particles inside the spherical container is an example of the the non-holonomic constraint, which is required here to be discussed.

(2) Scleronomic and Rheonomic Constraints:-
If the constraint are independent of time, they are termed as Scleronomic constraints. However, if they contain time explicitly, they are called as Rheonomic constraints.
Example:- (i) The simple pendulum with rigid support is an example of Scleronomic constraint. However, it is necessary to discuss here that why does the simple pendulum explain scleronomic constraint.

(ii) The motion of deformable body is an example of Rheonomic constraint. However, it is necessary to discuss here.

(3) Conservative System :-

The total mechanical energy of system is conserved while performing the constrained motion. In this type of constraint, the constrained forces do not do any work. The motion of rigid body is an example of conservative system, which is required here to be discussed.

(4) Dissipative System :- The constrained forces do work and total mechanical energy is not conserved. The pedulum with variable length is an example of this type of constraint, which is necessary to be discussed here.

(5) Bilateral and Unilateral systems :-

If at any point on the constrained surface both the forward and backward motion is possible, such a system is called the bilateral. In this system, the constraint relations are in the form of equations. The simple pendulum is an example of this type system, which is required here to be discussed.

However, if at some point of constrained surface, the forward motion is not possible and the constraint relations are expressed in the form of inequalities, then such a system is called unilateral. The motion of gas particle inside the spherical container of fixed radius is an example of this system. (Discussed).

3. Derive the Euler Lagrange's equations of motion from D'Alembert Principle for a holonomic conservative system.

Ans. :- We consider a rheonomic, holonomic system having 'N' particle and 'k' holonomic constraint. Thus it has $n = 3N-k$ degree of freedom. We choose $n = 3N-k$ generalize co-ordinate $(q_1, q_2 \cdots q_n)$ in such a manner that

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, \cdots q_n, t) \quad -(1)$$

$(i = 1, 2, 3, \cdots N)$

Where $\vec{r}_i$ be the position vector of a i^{th} particle at time 't' with respect to origin. Let us also assume that the mass of each particle be 'm_i'.

From eq. (1), $\vec{v}_i$ is expressed as

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \quad -(2)$$

where q_j is the generalize Co-ordinate and $\dot{q}_j$ is the generalize velocity.

Similarly, the arbitrary virtual displacement $\delta\vec{r}_i$ can be expressed in terms of δq_j by

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j + \frac{\partial \vec{r}_i}{\partial t} \delta t$$

No variation of time 'δt' is involved, since a virtual displacement by definition considers only displacement of the coordinates.

(5)

Therefore,

$$\delta \vec{r_i} = \sum_j \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j \quad \text{— (3)}$$

In terms of generalized co-ordinates, the virtual work done by $\vec{F_i}$ is expressed as

$$\sum_i \vec{F_i} \cdot \delta \vec{r_i} = \sum_{i,j} \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j$$

$$\sum_i \vec{F_i} \cdot \delta \vec{r_i} = \sum_j Q_j \delta q_j \quad -(4)$$

where Q_j's are called the component of the generalized fore, defined as

$$Q_j = \sum_i \vec{F_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \quad \text{—(5)}$$

From D'Alembert Principle, we have

$$\sum_{i=1}^{N} \left(\vec{F_i}^{(a)} - \dot{\vec{p_i}} \right) \cdot \delta \vec{r_i} = 0 \quad -(6)$$

$$\sum_{i=1}^{N} \left(\vec{F_i}^{(a)} - \dot{\vec{p_i}} \right) \cdot \left(\sum_j \frac{\partial \vec{r_i}}{\partial q_j} \delta q_j \right) = 0$$

$$\sum_{i=1}^{N} \vec{F_i}^{(a)} \cdot \sum_j \left(\frac{\partial \vec{r_i}}{\partial q_j} \delta q_j \right) - \sum_{i=1}^{N} \dot{\vec{p_i}} \cdot \sum_j \left(\frac{\partial \vec{r_i}}{\partial q_j} \delta q_j \right) = 0$$

$$\sum_{j=1}^{n} Q_j \delta q_j - \sum_{i=1}^{N} \dot{\vec{p_i}} \cdot \sum_{j=1}^{n} \left(\frac{\partial \vec{r_i}}{\partial q_j} \delta q_j \right) = 0$$

$$\left\{ \text{Because } Q_j = \sum_{i=1}^{N} \vec{F_i}^{(a)} \frac{\partial \vec{r_i}}{\partial q_j} \quad j = 1, 2, \cdots n = 3N - K \right\}$$

$$\sum_{j=1}^{n} Q_j \delta q_j - \sum_{j=1}^{n} \left(\sum_{i=1}^{N} m_i \ddot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \right) \delta q_j = 0 \quad -(7)$$

Now consider 2nd term of eq.(7),

$$\sum_{i=1}^{N}\left(m_i\, \ddot{\vec{r}}_i\, \frac{\partial \vec{r}_i}{\partial q_j}\right) = \sum_i \left[\frac{d}{dt}\left(m_i\dot{\vec{r}}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i\dot{\vec{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right)\right] \quad —(8)$$

Differentiating both side of eq.(2) partially w.r.t $\dot{q}_k$, we have

$$\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k} \quad (k = 1,2,3\cdots n = 3N-k) \quad —(9)$$

Also

$$\frac{\partial \dot{\vec{r}}_i}{\partial q_k} = \sum_{j=1}^{n} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial q_k \partial t} \quad —(10)$$

Therefore

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = \sum_{j=1}^{n} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial q_k \partial t} \quad —(11)$$

The kinetic energy of the system at time 't' is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_{i=1}^{N} m_i\, \dot{\vec{r}}_i\cdot\dot{\vec{r}}_i$$

$$\frac{\partial T}{\partial \dot{q}_j} = \frac{1}{2}\sum_{i=1}^{N}\left[m_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\cdot\dot{\vec{r}}_i + m_i\dot{\vec{r}}_i\cdot\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right]$$

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i\dot{\vec{r}}_i\cdot\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} \quad —(12)$$

From eq.(9), eq.(12) becomes,

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i\dot{\vec{r}}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j} \quad —(13)$$

similarly,

$$\frac{\partial T}{\partial q_j} = \frac{1}{2}\sum_{i=1}^{N}\left[m_i\dot{\vec{r}}_i\cdot\frac{\partial \dot{\vec{r}}_i}{\partial q_j} + m_i\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\cdot\dot{\vec{r}}_i\right]$$

$$\frac{\partial T}{\partial q_j} = \sum_{i=1}^{N} m_i\dot{\vec{r}}_i\cdot\frac{\partial \dot{\vec{r}}_i}{\partial q_j} \quad —(14)$$

$$\frac{\partial T}{\partial q_j} = \sum_{i=1}^{N}\left[m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right] \quad \text{—(14)}$$

Using eqs (13) and (14), Eq.(8) becomes

$$\sum_{i=1}^{N}\left(m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \quad \text{—(15)}$$

Now by supplying eq(15) in eq.(7), we get

$$\sum_{j=1}^{n} Q_j \, \delta q_j - \sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j = 0$$

$$\sum_{j=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right]\delta q_j = 0 \quad \text{-(16)}$$

Since the system is holonomic, $\delta q_1, \delta q_2 \cdots \delta q_n$ are linearly independent. Therefore, we have, from eq.(16),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{—(17)}$$

where $j = 1, 2, \cdots n = 3N - k$.

Eq.(17) is called 'Lagrange's equations for holonomic system.

If Q_j's are the functions of q_j's and 't' and not of generalized velocity, i.e.,

$$Q_j = Q_j(q_1, q_2 \cdots q_{n=3N-k}, t).$$

In such case, there exists an ordinary potential energy function $V(q_1, q_2 \cdots q_n, t)$ such that

$$Q_j = -\frac{\partial V}{\partial q_j}, \quad (j = 1, 2, 3, \cdots n = 3N-k) \qquad \text{— (18)}$$

Using eq.(18), eq.(17) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \qquad \text{— (19)}$$

We define a scalar function L, the Lagrangian as

$$L = T - V \qquad \text{— (20)}$$

where V is independent of $\dot{q}_1, \dot{q}_2 \cdots \dot{q}_n$.

Therefore, $$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \qquad \text{— (21)}$$

Eq.(19) becomes now

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \quad \text{where } L = T - V, \qquad \text{— (22)}$$

$L = L(q_1, q_2 \cdots q_n, \dot{q}_1, \dot{q}_2 \cdots \dot{q}_n, t)$ is the Lagrangian system. Equations (22) are called the Euler-Lagrange's equations of motion for holonomic and conservative system.

— x —

(9)

4. Explain the generalized co-ordinates and momenta and prove that
$$\sum_j \dot{q}_j P_j = 2T$$

Ans:- Generalize co-ordinates :- For a system with 'k' independent constraints, the number of independent variables required to specify its configuration and position is $n = 3N - k$, which is less than total number of Cartesian co-ordinates involved. We can choose any set of the required number of independent quantities say $(q_1, q_2, q_3 \cdots q_{n=3N-k})$ such that all Cartesian co-ordinates are known function of the q_i variables, i.e.,

$$x_1 = x_1(q_1, q_2 \cdots q_n, t)$$
$$x_2 = x_2(q_1, q_2 \cdots q_n, t)$$
$$\vdots$$
$$x_{3N} = x_{3N}(q_1, q_2 \cdots q_n, t) \quad -(1)$$

Thus, the generalized co-ordinates can be defined as the minimum number of independent co-ordinates required to completely describe the configuration of a dynamical system.
The inverse transformation equations can be written as

$$q_1 = q_1(x_1, y_1, z_1, x_2, y_2, z_2 \cdots x_N, y_N, z_N)$$
$$q_2 = q_2(x_1, y_2, z_1, x_2, y_2, z_2, \cdots x_N, y_N, z_N)$$
$$\vdots$$
$$q_i = q_i(x_1, y_1, z_1, x_2, y_2, z_2, \cdots x_N, y_N, z_N) \quad -(2)$$

Where $i = 1, 2, 3 \cdots n = 3N-k$. The time derivative of q_i, i.e., $\frac{dq_i}{dt} = \dot{q}_i$ is called the generalized velocity.

Generalized Momenta :- In the Lagragian dynamics, it would be proper to define both momentum and energy in terms of the given Lagrangian.

We know that the Newtonian momentum is defined as

$$P_i = m_i v_i = m_i \dot{x}_i \quad — (3)$$

The kinetic energy T is given by

$$T = \frac{1}{2} m_i \dot{x}_i^2 \quad — (4)$$

Therefore, $\frac{\partial T}{\partial \dot{x}_i} = \frac{1}{2} m_i 2\dot{x}_i = m_i \dot{x}_i$

$$P_i = \frac{\partial T}{\partial \dot{x}_i} \quad —(5)$$

If the potential energy part of 'L' is independent of both $\dot{x}_i$'s and 't', then p_i can be expressed in terms of 'L', i.e.;

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i}$$

[Because $\frac{\partial V}{\partial \dot{x}_i} = 0$].

Thus, we have

$$P_i = \frac{\partial L}{\partial \dot{x}_i} \quad — (6)$$

By using eq.(6), we understand that the generalized momentum p_i corresponding to a generalized co-ordinate q_i is defined as

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}} \quad — (7)$$

Since $p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$ —(8)

[Because $\frac{\partial V}{\partial \dot{q}_i} = 0$]

We know that $T = T_0 + T_1 + T_2$ where

$$T_0 = \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t}\right)^2 = a_0$$

$$T_1 = m_i \left(\frac{\partial r_i}{\partial t}\right)\left(\frac{\partial r_i}{\partial q_j}\right)\dot{q}_j = a_j \dot{q}_j$$

$$T_2 = \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial q_j}\right)\left(\frac{\partial r_i}{\partial q_k}\right)\dot{q}_j \dot{q}_k = \frac{1}{2} a_{jk} \dot{q}_j \dot{q}_k$$

$$T = a_0 + a_j \dot{q}_j + \frac{1}{2} a_{jk} \dot{q}_j \dot{q}_k \quad -(10) \quad -(9)$$

From eq. (9), we know that

$$\frac{\partial T_0}{\partial \dot{q}_j} = 0$$

From eq. (8), we can have

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial (T_0 + T_1 + T_2)}{\partial \dot{q}_j}$$

or

$$p_j = \frac{\partial (T_1 + T_2)}{\partial \dot{q}_j} \quad —— (11)$$

Now multiply $\dot{q}_j$ on the both side of eq. (11), we get

$$\dot{q}_j p_j = \dot{q}_j \frac{\partial T_1}{\partial \dot{q}_j} + \dot{q}_j \frac{\partial T_2}{\partial \dot{q}_j} \quad —(12)$$

If f is a homogeneous function of order 'n', of a set of variable q_j, then

$$\boxed{\sum_j q_j \frac{\partial f}{\partial q_j} = nf}$$

$\dot{q}_j \frac{\partial T_1}{\partial \dot{q}_j} = T_1$ [T_1 is expressed in terms of a term linear in generalized velocity]

$\dot{q}_j \frac{\partial T_2}{\partial \dot{q}_j} = 2T_2$ [T_2 is expressed in terms of a term that is quadratic in generalize velocity]

Therefore, eq. (12) becomes

$\dot{q}_j p_j = [T_1 + 2T_2] = 2T - T_1 - 2T_0$ — (13)

For scleronomic system having ordinary potential forces, $T_1 = T_0 = 0$.

Thus, we get

$$\dot{q}_j p_j = 2T \quad \text{— (14)}$$

or

For all independent $\dot{q}_j$'s where $j = 1, 2 \cdots n$,

$$\boxed{\sum_j \dot{q}_j p_j = 2T}$$

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5. Show that under the Lagrangian gauge transformation, Euler Lagrange's equations of motion remain invariant.

Ans. :- Euler-Lagrange's equations do not change if we add to the Lagrangian a total time derivative of any arbitrary function $F(q_j, t)$. Such a function is called Gauge function for Lagrangian. It means that

$L' = L(q_j, \dot{q}_j, t) + \frac{dF(q_j, t)}{dt}$ also satisfies the Lagrange's equation of motion.

The Euler Lagrange's equations of motion can be written as $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$ —(1)

Now we replace L by $L' = L + \frac{dF}{dt}$ in eq. (1), we get

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(L + \frac{dF}{dt}\right)\right] - \frac{\partial\left(L + \frac{dF}{dt}\right)}{\partial q_j} =$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial (dF/dt)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}\left(\frac{dF}{dt}\right)$$

$=$ From eq. (1), We have

$$= \frac{d}{dt}\left[\frac{\partial (dF/dt)}{\partial \dot{q}_j}\right] - \frac{\partial}{\partial q_j}\left(\frac{dF}{dt}\right)$$

So we get

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(L + \frac{dF}{dt}\right)\right] - \frac{\partial\left(L + \frac{dF}{dt}\right)}{\partial q_j} = \frac{d}{dt}\left[\frac{\partial (dF/dt)}{\partial \dot{q}_j}\right] - \frac{\partial}{\partial q_j}\left(\frac{dF}{dt}\right) \quad \text{— (2)}$$

since $F = F(q_j, t)$, we get

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t} \quad \text{— (3)}$$

$$\frac{d}{dt}\left(\frac{\partial (dF/dt)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}\left(\frac{dF}{dt}\right) = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i}\dot{q}_i\right)\right] - \frac{\partial}{\partial q_j}\left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i}\dot{q}_i\right]$$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{q}_j}\right) + \dot{q}_i \frac{\partial}{\partial q_i}\left(\frac{\partial F}{\partial \dot{q}_j}\right) + \frac{\partial F}{\partial q_i}\frac{\partial \dot{q}_i}{\partial \dot{q}_j}\right] - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial q_i}\right)\dot{q}_i - \frac{\partial F}{\partial q_i}\frac{\partial \dot{q}_i}{\partial q_j}$$

As F is not a function of $\dot{q}_j$, $\frac{\partial F}{\partial \dot{q}_j} = 0$. Also $\frac{\partial \dot{q}_i}{\partial q_j} = 0$

and $\frac{\partial \dot{q}_i}{\partial \dot{q}_j} = \delta_{ij} = 1$ if $i = j$.

Therefore for above equation,

$$\text{R.H.S} = \frac{d}{dt}\left(\frac{\partial F}{\partial q_j}\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial q_i}\right)\dot{q}_i$$

$$\text{R.H.S} = \frac{\partial}{\partial q_j}\left[\frac{dF}{dt}\right] - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial q_i}\right)\dot{q}_i$$

$$R.H.S. = \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial q_i}\right)\dot{q}_i$$

$$R.H.S. = \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial q_i}\right)\dot{q}_i + \frac{\partial F}{\partial q_i}\frac{\partial \dot{q}_i}{\partial q_j} + \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right) - \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial q_i}\right)\dot{q}_i$$

As $\frac{\partial \dot{q}_i}{\partial q_j} = 0$, therefore

$R.H.S = 0$

Thus we get

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = 0, \quad \text{where } L' = L + \frac{dF}{dt}. \quad \text{— (4)}$$

and $F(q_j, t)$ is called the gauge function for the Lagrangian.

From eq. (4), we understand that under Lagrangian gauge transformation, the Euler Lagrange's equations of motion remain invariant.

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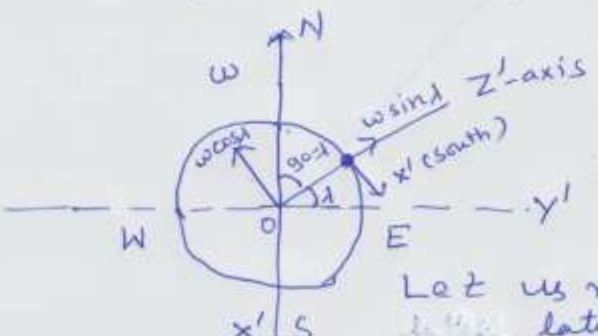
6. What is Coriolis force? Discuss the effect of Coriolis force on the free fall of a body on earth's surface.

Ans:- Coriolis Force :- The coriolis force acts on a particle when it moves relative to the rotating frame (non-inertial frame) and it's directly proportional to the velocity of the particle. Also the Coriolis force is perpendicular to $\vec{\omega}$ and $\vec{v}$, i.e, angular velocity of rotating frame and velocity of the particle relative to rotating frame. Thus, the Coriolis force is defined as

$$\vec{F_c} = 2m\,\vec{v}\times\vec{\omega}.$$

Effect of Coriolis Force on the Free Fall of a Body on Earth's Surface.

The earth constitutes a rotating system with angular velocity $\omega = 7.292 \times 10^{-5}$ rad/s



Let us make observation at latitude λ. The earth rotates about north-south axis with angular speed ω. Let Z'-axis be taken in direction OP. X'-axis points towards South and Y'-axis point towards East. Since the particle moves towards the earth, its velocity in rotating system $\vec{v}$ is not zero. We also neglect the variation of $\vec{g}$ with latitude. Therefore, the acceleration of the body at any place on the earth is given by

$$\dot{\vec{v}} = 2\vec{v}\times\vec{\omega} + (\vec{\omega}\times\vec{r})\times\vec{\omega} + \vec{r}\times\dot{\vec{\omega}} - \frac{\vec{\nabla}V}{m}$$

where $-\vec{\nabla}V = m\vec{g}$ ($\vec{g}$ = acceleration purely due to earth's gravity)

$$\dot{\vec{v}} = 2\vec{v}\times\vec{\omega} + (\vec{\omega}\times\vec{r})\times\vec{\omega} + \vec{r}\times\dot{\vec{\omega}} + \vec{g} \quad -(1)$$

In eq (1), $\vec{\omega}$ is the constant angular velocity of earth at the point under consideration. Therefore,

$$\dot{\vec{\omega}} = \frac{d\vec{\omega}}{dt} = 0$$

Thus eq. (1) becomes,

$$\dot{\vec{v}} = 2\vec{v}\times\vec{\omega} + (\vec{\omega}\times\vec{r})\times\vec{\omega} + \vec{g} \quad -(2)$$

We define $\vec{g}_{eff} = (\vec{\omega}\times\vec{r})\times\vec{\omega} + \vec{g} \quad -(3)$

By using eq.(3), eq.(2) becomes

$$\dot{\vec{v}} = 2\vec{v}\times\vec{\omega} + \vec{g}_{eff} \quad —(4)$$

Now the equation of motion of the body is

$$m\dot{\vec{v}} = 2m\,\vec{v}\times\vec{\omega} + m\vec{g}_{eff} \quad —(5)$$

Now
$$\left.\begin{aligned}\vec{v} &= (i'\dot{x}' + j'\dot{y}' + k'\dot{z}')\\ \vec{\omega} &= (i'w_x' + j'w_y' + k'w_z')\end{aligned}\right\} —(6)$$

From the above Figure, we have

$$\left.\begin{aligned}w_x' &= -w\cos\lambda\\ w_y' &= 0\\ w_z' &= \omega\sin\lambda\end{aligned}\right\} —(7)$$

Therefore, the equation of motion will be

$$m\dot{\vec{v}} = m\vec{g}_{eff} - 2m\left[(i'w_x' + j'w_y' + k'w_z')\times(i'\dot{x}' + j'\dot{y}' + k'\dot{z}')\right]$$

$$m\dot{\vec{v}} = m\vec{g}_{eff} - 2m\left[(i'w_x' + k'w_z')\times(i'\dot{x}' + j'\dot{y}' + k'\dot{z}')\right]$$

$$m\frac{d^2x'}{dt^2}i' + m\frac{d^2y'}{dt^2}j' + m\frac{d^2z}{dt^2}k' = -(mg_{eff} + 2mw_x'\dot{y}')k' - 2m(w_z'\dot{x}' - w_x'\dot{z}')j' + i'2mw_z'\dot{y}'$$

Thus equations which govern the free fall are (8)

$$m\frac{d^2x'}{dt^2} = 2mw_z'\dot{y}' = 2m\omega\sin\lambda\;\dot{y}' = 2m\omega\sin\lambda\frac{dy'}{dt} \quad —(9)$$

$$m\frac{d^2y'}{dt^2} = -2m(w_z'\dot{x}' - w_x'\dot{z}')$$

$$m\frac{d^2y}{dt^2} = -2m\frac{dx'}{dt}\omega\sin\lambda - 2m\frac{dz'}{dt}\omega\cos\lambda \quad —(10)$$

$$m\frac{d^2z}{dt^2} = -mg_{eff} + 2m\omega\cos\lambda\frac{dy'}{dt} \quad —(11)$$

We have taken $(-mg_{eff})$ because direction of $\vec{g}$ and $+z'$-axis are opposite.

(17) Since the body falls under the action of gravity the velocity will be almost along z'-axis. Thus we can neglect $\frac{dy'}{dt}$ and $\frac{dx'}{dt}$.

After neglecting $\frac{dx'}{dt}$ and $\frac{dy'}{dt}$, we get

$$m\frac{d^2x'}{dt^2} = 0 \quad -(12)$$

$$m\frac{d^2y'}{dt^2} = -2m\omega\cos\lambda\frac{dz'}{dt} \quad -(13)$$

$$m\frac{d^2z'}{dt^2} = -mg_{eff} \quad -(14)$$

Eq(12) gives $\frac{dx'}{dt}$ = constant, which we have assumed as negligible, i.e., there is no deviation in the North-South direction.

From eq.(14), we get

$$\frac{dz'}{dt} = -g_{eff}\,t \quad -(15)$$

(Constant of integration is zero as body starts initially from rest.)

putting $\frac{dz'}{dt} = -g_{eff}\,t$ in eq. (13), we get

$$\cancel{m}\frac{d^2y'}{dt^2} = +2\cancel{m}\omega g_{eff}\,t\cos\lambda$$

$$\frac{d^2y'}{dt^2} = 2\omega g_{eff}\,t\cos\lambda \quad -(16)$$

Integrating twice and taking $\left(\frac{dy'}{dt}\right)_{t=0} = 0$ and $(y')_{t=0} = 0$ we get

$$y' = \frac{1}{3}\omega g_{eff}\,t^3\cos\lambda \,. \quad -(17)$$

If h be the height of the free fall, the time spent in flight will be $t = \sqrt{(2h/g_{eff})}$ because $h = \frac{1}{2}g_{eff}t^2$. $-(18)$

Suppose body falls from 100 meters above the ground then

$$t^2 = \frac{2 \times 100 \times 10^2}{981} = 20 \text{ sec}^2$$

Now we know that $\omega \cong 7 \times 10^{-5}$ rad/s —(19)

Therefore, the displacement at the equator ($\lambda = 0$) will be

$$y' = \frac{1}{3} \omega g_{eff} t^3 \cong 3 \text{cm}, \quad —(20)$$

which is difficult to be detected since positive y'-axis point towards east; hence the body deflects by 3 cm towards east or y'-axis direction, which is perpendicular to both $\vec{\omega}$ (NS-axis) and $\vec{v}$ (z'-axis). Consequently this displacement is due to Coriolis force (which is perpendicular to both $\vec{\omega}$ and $\vec{v}$).

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7. Write the Hamiltonian function and the equations of motion for a two dimentional isotropic harmonic oscillator using the Cartesian coordinates.

Ans :- The Lagrangian function is

$$L = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k(x^2 + y^2)$$

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k(x^2 + y^2) \quad —(1)$$

giving

$$\left. \begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{or} \quad \dot{x} = \frac{p_x}{m} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} \quad \text{or} \quad \dot{y} = \frac{p_y}{m} \end{aligned} \right\} —(2)$$

The Hamiltonian H is given by

$$H = \sum_j p_j \dot{q}_j - L$$

$$H = p_x \dot{x} + p_y \dot{y} - \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k(x^2 + y^2)$$

$$H = \frac{p_x^2}{m} + \frac{p_y^2}{m} - \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} k(x^2 + y^2)$$

$$H = \frac{p_x^2}{m} + \frac{p_y^2}{m} - \frac{1}{2} m\left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2}\right) + \frac{1}{2} k(x^2 + y^2)$$

$$H = \frac{1}{2m}(p_x^2+p_y^2) + \frac{1}{2}k(x^2+y^2) \quad \text{— (3)}$$

giving

$$\left.\begin{aligned} \frac{\partial H}{\partial p_x} &= \frac{p_x}{m}, \quad \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \\ \frac{\partial H}{\partial x} &= kx, \quad \frac{\partial H}{\partial y} = ky \end{aligned}\right\} \text{— (4)}$$

The Hamiltonian's equation are

$$\left.\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = -kx \end{aligned}\right\} \text{— (5)}$$

and

$$\left.\begin{aligned} \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \\ \dot{p}_y &= -\frac{\partial H}{\partial y} = -ky \end{aligned}\right\} \text{— (6)}$$

It is quite easy to show from eq. (5) and eq (6), that

$$\left.\begin{aligned} m\ddot{x} + kx &= 0 \\ m\ddot{y} + ky &= 0 \end{aligned}\right\} \text{— (7)}$$

which are the desired equation of motion of two dimentional isotropic harmonic oscillator in Cartesian co-ordinates.

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8. Show that the transformation

$$Q = \sqrt{2q}\, e^{a} \cos p, \quad P = \sqrt{2q}\, e^{-a} \sin p$$

is canonical.

<u>Ans.</u>:- If transformation is canonical, then $(PdQ - pdq)$ would be an exact differential.

$$dQ = \frac{1}{\cancel{2}}(2q)^{-1/2}\cancel{2}\,dq\, e^{a}\cos p - (2q)^{1/2} e^{a} \sin p\, dp. \qquad -(1)$$

Therefore,

$$(PdQ - pdq) = P(2q)^{-1/2} dq\, e^{a}\cos p - P(2q)^{1/2} e^{a}\sin p\, dp - pdq$$

$$(PdQ - pdq) = \left[\cancel{(2q)^{1/2}}\, \cancel{e^{-a}} \sin p\, \cancel{(2q)^{-1/2}}\, \cancel{e^{a}} \cos p\, dq - (2q)^{1/2} \cancel{e^{-a}} \sin p\,(2q)^{1/2} \cdot \cancel{e^{a}} \sin p\, dp - pdq\right]$$

$$(PdQ - pdq) = \sin p \cos p\, dq - 2q \sin^2 p\, dp - pdq$$

$$(PdQ - pdq) = (\sin p \cos p - p)\,dq - 2q \sin^2 p\, dp$$

$$(PdQ - pdq) = \frac{\partial}{\partial q}\left[\frac{1}{2} q \sin 2p - pq\right] dq + \frac{\partial}{\partial p}\left[\frac{1}{2} q \sin 2p - pq\right] dp$$

$$(PdQ - pdq) = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial p} dp$$

$$(pdQ - pdq) = dF \qquad -(2)$$

Which shows that right-hand side is an exact differential of the function

$$F = \left(\frac{1}{2} q \sin 2p - pq\right)$$ and hence,

the transformation is canonical.

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